

VIRTUAL POINCARÉ POLYNOMIAL OF THE SPACE OF STABLE PAIRS SUPPORTED ON QUINTIC CURVES

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ABSTRACT. Let $\mathbf{M}^\alpha(d, \chi)$ be the moduli space of α -stable pairs (s, F) on the projective plane \mathbb{P}^2 with Hilbert polynomial $\chi(F(\mathbf{m})) = d\mathbf{m} + \chi$. For sufficiently large α (denoted by ∞), it is well known that the moduli space is isomorphic to the relative Hilbert scheme of points over the universal degree d plane curve. For the general (d, χ) , the relative Hilbert scheme does not have a bundle structure over the Hilbert scheme of points. In this paper, as the first non trivial such a case, we study the wall crossing of the α -stable pairs space when $(d, \chi) = (5, 2)$. As a direct corollary, by combining with Bridgeland wall crossing of the moduli space of stable sheaves, we compute the virtual Poincaré polynomial of $\mathbf{M}^\infty(5, 2)$.

1. INTRODUCTION

1.1. Introduction and results. By definition, a pair (s, F) consists of a sheaf F on \mathbb{P}^2 and one-dimensional subspace $s \subset H^0(F)$. Let us fix $\alpha \in \mathbb{Q}[\mathbf{m}]$ with a positive leading coefficient. A pair (s, F) is called α -*semistable* if F is pure and for any subsheaves $F' \subset F$, the inequality

$$\frac{\chi(F'(\mathbf{m})) + \delta \cdot \alpha}{r(F')} \leq \frac{\chi(F(\mathbf{m})) + \alpha}{r(F)}$$

holds for $\mathbf{m} \gg 0$. Here $r(F)$ is the leading coefficient of the Hilbert polynomial $\chi(F(\mathbf{m}))$ and $\delta = 1$ if the section s factors through F' and $\delta = 0$ otherwise. When the strict inequality holds, we say (s, F) is α -stable.

With the help of the general result of the geometric invariant theory ([19]), Le Potier ([17, Theorem 4.12]) proved that there exist projective schemes $\mathbf{M}^\alpha(d, \chi)$ parameterizing α -stable pairs (s, F) such that F has Hilbert polynomial $P(\mathbf{m}) = d\mathbf{m} + \chi$. Also, M. He ([13]) studied the wall crossings (or flips) of the moduli spaces $\mathbf{M}^\alpha(d, \chi)$ as α varies. In two extremal case,

- If $\deg(\alpha) \geq 2$, then $\mathbf{M}^{\alpha:=\infty}(d, \chi)$ is isomorphic to the relative Hilbert scheme of $\mathbf{n} = \chi - \frac{d(3-d)}{2}$ points on the universal degree d curve ([13, §4.4], [22, Proposition B.8]). Let us denote by $\mathbf{B}(d, \mathbf{n})$ the relative Hilbert scheme. When $\alpha = \infty$, α -stable pairs are precisely stable pairs in the sense of Pandharipande-Thomas ([22]).
- If α is sufficiently small (denoted by $\alpha = +$), the moduli space has a natural forgetful morphism

$$\xi : \mathbf{M}^+(d, \chi) \longrightarrow \mathbf{M}(d, \chi)$$

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which associates to the 0^+ -stable pair (s, F) the sheaf F . The later moduli space $\mathbf{M}(\mathbf{d}, \chi)$ parameterizes S -equivalent classes¹ of semistable sheaves with Hilbert polynomial $\mathbf{d}m + \chi$ ([14]).

When $\chi = 1$, the moduli space $\mathbf{M}(\mathbf{d}, 1)$ is a smooth projective variety of dimension $\mathbf{d}^2 + 1$. Relating with the curve counting invariants on the (open) Calabi-Yau threefolds, S. Katz ([15]) conjectured the signed topological Euler number

$$(-1)^{\mathbf{d}^2+1} \cdot e(\mathbf{M}(\mathbf{d}, 1)) = \mathbf{n}_{0, \mathbf{d}}$$

is exactly the genus 0, BPS (or GV)-number $\mathbf{n}_{0, \mathbf{d}}$ on the local \mathbb{P}^2 (i.e., the total space of the canonical line bundle of \mathbb{P}^2). For the introduction of these subjects, see [23]. For the local \mathbb{P}^2 , several authors confirmed that this conjecture holds ([24, 27, 5, 7, 6]) for the lower degree cases through several different methods. Specially, in [5], when $\mathbf{d} \leq 5$ and $\chi = 1$, the authors show that the moduli spaces $\mathbf{M}^\alpha(\mathbf{d}, 1)$ are birational among each other and thus we obtain the cohomology group of the space $\mathbf{M}(\mathbf{d}, 1)$ by studying the wall crossing of the moduli spaces $\mathbf{M}^\alpha(\mathbf{d}, 1)$. In this case, the work is well-going since the relative Hilbert scheme $\mathbf{B}(\mathbf{d}, \mathbf{n})$ has a projective bundle structure and all of the wall crossings are *simple*, that is, the length of the JH-filtrations of α -stable pairs is two. But, for the large (\mathbf{d}, χ) , the wall crossings among the α -stable pairs space become very complicate because the wall crossing may not be simple. Also it is hard to understand the geometry of the relative Hilbert scheme $\mathbf{B}(\mathbf{d}, \mathbf{n})$ (cf. [5]). Hence we need more careful study to get some geometric information of the space $\mathbf{M}(\mathbf{d}, \chi)$ from the α -stable pairs spaces or its wall crossings. In this paper, we study the wall crossings when $(\mathbf{d}, \chi) = (5, 2)$, which is the first case such that the relative Hilbert scheme is not a projective bundle and the wall crossings may not be simple. That is, we will show that

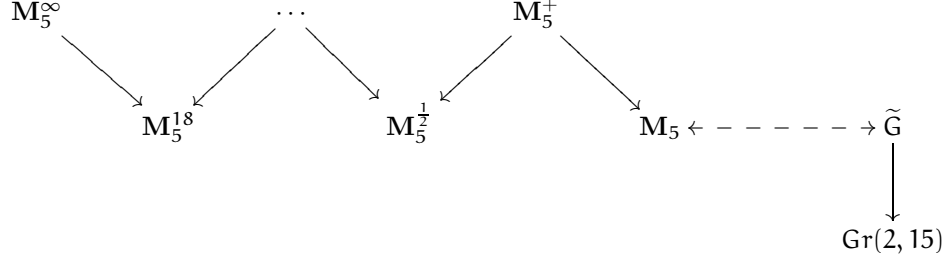
Theorem 1.1. (1) *There are five wall crossings between $\mathbf{M}^\infty(5, 2)$ and $\mathbf{M}^+(5, 2)$; the walls occur at $\alpha = 18, 13, 8, 3$ and $\frac{1}{2}$.*
 (2) *The forgetful map $\mathbf{M}^+(5, 2) \rightarrow \mathbf{M}(5, 2)$ is a projective bundle on $\mathbf{M}^+(5, 2)_2$ with fiber \mathbb{P}^1 . In the complement of $\mathbf{M}^+(5, 2)_2$, it is a \mathbb{P}^2 -bundle map.*
Here $\mathbf{M}^+(5, 2)_2$ is the locus of 0^+ -stable pairs (s, F) with $h^0(F) = 2$.

On the other hand, the moduli space $\mathbf{M}(5, 2)$ has another wall crossings, that is, the Bridgeland wall crossing. This was done in a general setting by many authors (for example, [26, 2]). In order to get the cohomology group of the space $\mathbf{M}(5, 2)$ from the Bridgeland wall crossing as it was done in [6], it is essential to know the final birational (or wall crossing model) of the moduli space $\mathbf{M}(5, 2)$. By studying the nef cone of one of the birational model of $\mathbf{M}(5, 2)$, we obtain

Proposition 1.2. *The final birational model of $\mathbf{M}(5, 2)$ is isomorphic to the Grassmannian variety $\mathrm{Gr}(2, 15)$.*

¹Two semistable sheaves are S -equivalent if they have isomorphic Jordan-Hölder filtration.

The wall crossings of two different types are summarized into the following diagram. Let $\mathbf{M}^\alpha(5, 2) := \mathbf{M}_5^\alpha$ and $\mathbf{M}(5, 2) := \mathbf{M}_5$.



As a direct corollary,

Corollary 1.3. *The virtual Poincaré polynomial of the space $\mathbf{M}^\infty(5, 2)$ is given by*

$$\begin{aligned} &1 + 3p + 9p^2 + 22p^3 + 50p^4 + 99p^5 + 173p^6 + 256p^7 + 330p^8 + 379p^9 + 407p^{10} \\ &+ 420p^{11} + 426p^{12} + 428p^{13} + 429p^{14} + 428p^{15} + 423p^{16} + 410p^{17} + 382p^{18} \\ &+ 333p^{19} + 259p^{20} + 176p^{21} + 101p^{22} + 51p^{23} + 22p^{24} + 9p^{25} + 3p^{26} + p^{27}. \end{aligned}$$

Remark 1.4. In particular, the virtual Euler number of $\mathbf{M}^\infty(5, 2)$ is $e(\mathbf{M}^\infty(5, 2)) = 6030$. But the virtual Euler number of the PT-space of local \mathbb{P}^2 (that is, the total space of the canonical line bundle $K_{\mathbb{P}^2}$) is 6060; this is obtained using the torus localization technique ([3]). The difference 30 comes from the Euler number of the sheaves supported on \mathbb{P}^1 which is the complement of the zero section of local \mathbb{P}^2 . This was reported to the author by J. Choi. The author would like to thank J. Choi for the comment.

1.2. Stream of the paper. In §2, we study the wall crossing of the moduli spaces of α -stable pairs on \mathbb{P}^2 by using the classification of semistable sheaves in [9, 18]. Also, we analyze the forgetful map ξ by considering the Brill-Noether locus in $\mathbf{M}(5, 2)$. In §3, we find the last birational model of $\mathbf{M}(5, 2)$ by studying the effective cone of the moduli space $\mathbf{M}(5, 2)$. As a corollary, we obtain the Poincaré polynomial of the space $\mathbf{M}(5, 2)$ which reprove the result of [27]. In §4, we compute the Poincaré polynomial of the relative Hilbert scheme $\mathbf{B}(5, 7)$ by using the result of the previous sections.

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2. WALL CROSSINGS OF THE SPACES $\mathbf{M}^\alpha(5, 2)$

In this section, we firstly study the wall crossing among $\mathbf{M}^\infty(5, 2)$ and $\mathbf{M}^+(5, 2)$. Secondly, we analyze the forgetful map $\mathbf{M}^+(5, 2) \rightarrow \mathbf{M}(5, 2)$ defined in the introduction by analyzing the Brill-Noether locus. For convenience of the reader, we state the following useful results which will be used several times in this paper.

Lemma 2.1. [13, Corollary 1.6] *Let $\Lambda = (s, F)$ and $\Lambda' = (s', F')$ be pairs on a smooth projective variety X . There exists a long exact sequence*

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}(\Lambda, \Lambda') &\rightarrow \operatorname{Hom}(F, F') \rightarrow \operatorname{Hom}(s, H^0(F')/s') \\ &\rightarrow \operatorname{Ext}^1(\Lambda, \Lambda') \rightarrow \operatorname{Ext}^1(F, F') \rightarrow \operatorname{Hom}(s, H^1(F')) \\ &\rightarrow \operatorname{Ext}^2(\Lambda, \Lambda') \rightarrow \operatorname{Ext}^2(F, F') \rightarrow \operatorname{Hom}(s, H^2(F')) \rightarrow \cdots \end{aligned}$$

On the other hand, let X be a quasi-projective variety. Let us denote by

$$P(X) = \sum_i (-1)^i \dim H^i(X) p^i$$

the *virtual* Poincaré polynomial of X . Let $e(X) := \sum_i (-1)^i \dim H^i(X)$ be the *virtual* Euler number of the variety X . The virtual Poincaré polynomial has the following *motivic* properties.

Proposition 2.2. (1) $P(X) = P(X - Z) + P(Z)$ for a closed subvariety Z of X .
(2) Let X and Y be quasi-projective varieties. Let $\pi : X \rightarrow Y$ be a Zariski locally trivial fibration with fiber F . Then $P(X) = P(Y) \cdot P(F)$.
(3) Let $f : X \rightarrow Y$ be a bijective morphism. Then $P(X) = P(Y)$.
In (2), if the fiber is $F \cong \operatorname{Gr}(k, n)$, the same conclusion holds even though π is an analytic fibration ([1, Lemma 3.1]).

2.1. Wall crossing between $M^\infty(5, 2)$ and $M^+(5, 2)$. The possible types of strictly semistable pairs are given in the following table.

$(d, \chi) = (5, 2)$	
α	Types of the JH-filtration of the pair $(1, (5, 2))$ at α
18	$(1, (4, -2)) \oplus (0, (1, 4))$
13	$(1, (4, -1)) \oplus (0, (1, 3))$
8	$(1, (4, 0)) \oplus (0, (1, 2))$
3	$(1, (4, 1)) \oplus (0, (1, 1))$
3	$(1, (3, 0)) \oplus (0, (2, 2))$
3	$(1, (3, 0)) \oplus (0, (1, 1)) \oplus (0, (1, 1))$
$\frac{1}{2}$	$(1, (3, 1)) \oplus (0, (2, 1))$

Here $(1, (d, \chi))$ (resp. $(0, (d, \chi))$) denotes the pair (s, F) with a nonzero (resp. zero) section s and the Hilbert polynomial $\chi(F(m)) = dm + \chi$. All the wall crossings except at $\alpha = 3$ are simple. The wall occurs by following the configuration of points in quintic curves (Remark 2.11). In this subsection, we will describe the wall crossing for the computation of the virtual Poincaré polynomial of the space $M^\infty(5, 2)$.

Let us denote the C_α^+ (resp. C_α^-) by the wall crossing locus of the moduli space $M^{\alpha-\epsilon}(5, 2)$ (resp. $M^{\alpha+\epsilon}(5, 2)$) for sufficient small $\epsilon > 0$. During the following lemmas, we use that $M(1, \chi) \cong M(1, 1)$ by $F \mapsto F(-\chi + 1)$ and $M(1, 1) \cong \mathbb{P}^2$. Let us start with the study of the wall crossing at $\alpha = 18$. It turns out that the wall crossing locus C_{18}^- is *not* a projective bundle over its base space.

Lemma 2.3. *The wall crossing locus C_{18}^+ at $\alpha = 18$ is a \mathbb{P}^7 -bundle over the product space $\mathbb{P}^2 \times \mathbb{P}^{14}$. The locus C_{18}^- is a \mathbb{P}^3 -bundle over $\mathbb{P}^2 \times \mathbb{P}^{14} - D$ where $D = \mathbb{P}^2 \times \mathbb{P}^9$ and a \mathbb{P}^4 -bundle over D .*

Proof. By the analysis of the wall at $\alpha = 18$, the $18 + \epsilon$ -stable pairs $(1, F)$ in C_{18}^+ fits into a non-split exact sequence

$$(2.1) \quad 0 \rightarrow (0, F_{m+4}) \rightarrow (1, F) \rightarrow (1, F_{4m-2}) \rightarrow 0,$$

where $F_{dm+\chi}$ denotes any semistable sheaf with Hilbert polynomial $dm + \chi$. Also, one can easily check that all the pairs fitting in a non-split exact sequence as (2.1) are $\alpha + \epsilon$ -stable. Thus the wall C_{18}^+ is a $\mathbb{P}(\text{Ext}^1((1, F_{4m-2}), (0, F_{m+4})))$ -bundle over $\mathbf{M}^\infty(4, -2) \times \mathbf{M}(1, 4) \cong \mathbb{P}^{14} \times \mathbb{P}^2$. Here, $\mathbf{M}^\infty(4, -2) \cong \mathbf{B}(4, 0) = \mathbb{P}^{14}$ by [5, Lemma 2.3] and $\mathbf{M}(1, 4) \cong \mathbf{M}(1, 1) = \mathbb{P}^2$ by $F \mapsto F(-3)$. Let $\chi(F) = h^0(F) - h^1(F)$. Let $\chi(F, F') = \dim \text{Ext}^0(F, F') - \dim \text{Ext}^1(F, F') + \dim \text{Ext}^2(F, F')$. Since $\text{Ext}^0((1, F_{4m-2}), (0, F_{m+4})) = 0$ and $H^1(F_{m+4}) = 0$, from the exact sequence in Lemma 2.1, we obtain that

$$\dim \text{Ext}^1((1, F_{4m-2}), (0, F_{m+4})) = h^0(F_{m+4}) - \chi(F_{4m-2}, F_{m+4}).$$

Note that $F_{4m-2} \cong \mathcal{O}_C$ and $F_m \cong \mathcal{O}_L(3)$ for some quartic curve C and a line L . By using the resolution of F_{4m-2} , we obtain that $\dim \text{Ext}^1((1, F_{4m-2}), (0, F_{m+4})) = 8$.

Similar argument shows that the wall C_{18}^- is a $\mathbb{P}(\text{Ext}^1((0, F_{m+4}), (1, F_{4m-2})))$ -fibration over $\mathbf{M}^\infty(4, -2) \times \mathbf{M}(1, 4) \cong \mathbb{P}^{14} \times \mathbb{P}^2$. From the exact sequence in Lemma 2.1 again, one can see that

$$(2.2) \quad \text{Ext}^1((0, F_{m+4}), (1, F_{4m-2})) \cong \text{Ext}^1(F_{m+4}, F_{4m-2}).$$

From the short exact sequence $0 \rightarrow \mathcal{O}(2) \rightarrow \mathcal{O}(3) \rightarrow F_{m+4} \rightarrow 0$,

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(F_{m+4}, F_{4m-2}) &\rightarrow H^1(F_{4m-2}(-3)) \rightarrow H^1(F_{4m-2}(-2)) \rightarrow \\ &\text{Ext}^2(F_{m+4}, F_{4m-2}) \rightarrow 0. \end{aligned}$$

By Serre duality, $\text{Ext}^2(F_{m+4}, F_{4m-2}) \cong \text{Ext}^0(F_{4m-2}, F_{m+1})$. But the later space is zero if $L \not\subseteq C$ and \mathbb{C} otherwise. So we have,

$$\text{Ext}^1(F_{m+4}, F_{4m-2}) \simeq \begin{cases} \mathbb{C}^4 & \text{if } L \not\subseteq C, \\ \mathbb{C}^5 & \text{if } L \subseteq C. \end{cases}$$

Applying this fact in (2.2), we get the result. \square

Remark 2.4. The moduli space $\mathbf{M}^\infty(5, 2)$ is not smooth. In fact, let $(1, F)$ be a ∞ -stable pair fitting into a non-split exact sequence in (2.1) such that $F_{m+4} \cong \mathcal{O}_L(3)$ and $F_{4m-2} \cong \mathcal{O}_{C \cdot L}$ for some line L and cubic curve C . Applying the functor $\text{Ext}^\bullet(-, (1, F))$ (resp. $\text{Ext}^\bullet((0, F_{m+4}), -)$) to (2.1), we obtain

$$\text{Ext}^2((1, F_{4m-2}), (1, F)) \rightarrow \text{Ext}^2((1, F), (1, F)) \xrightarrow{a} \text{Ext}^2((0, F_{m+4}), (1, F))$$

(resp.

$$\text{Ext}^2((0, F_{m+4}), (0, F_{m+4})) \rightarrow \text{Ext}^2((0, F_{m+4}), (1, F)) \xrightarrow{b} \text{Ext}^2((0, F_{m+4}), (1, F_{4m-2}))).$$

By some diagram chasing and Lemma 2.1, one can check that the composition map

$$b \circ a : \text{Ext}^2((1, F), (1, F)) \rightarrow \text{Ext}^2((0, F_{m+4}), (1, F_{4m-2}))$$

is an isomorphism. The second term $\text{Ext}^2((0, F_{m+4}), (1, F_{4m-2})) \cong \text{Ext}^2(F_{m+4}, F_{4m-2})$ is isomorphic to \mathbb{C} by the proof of Lemma 2.3. Also $\text{Ext}^0((1, F), (1, F)) \cong \mathbb{C}$ by the ∞ -stability of the pair $(1, F)$. But $\chi((s, F), (s, F)) := \sum_i (-1)^i \dim \text{Ext}^i((s, F), (s, F)) = -26$ for all $(s, F) \in \mathbf{M}^\infty(5, 2)$ by Lemma 2.1 and [14, Lemma 6.13]. Therefore, we

have $\text{Ext}^1((1, F), (1, F)) \cong \mathbb{C}^{28}$. This implies that $\mathbf{M}^\infty(5, 2)$ is not smooth at $(1, F)$ because by [13, Lemma 4.10] we have

$$\dim_{(1, F)} \mathbf{M}^\infty(5, 2) = 27 < 28 = \dim T_{(1, F)} \mathbf{M}^\infty(5, 2).$$

Lemma 2.5. (1) *The wall crossing locus C_{13}^+ (resp. C_{13}^-) at $\alpha = 13$ is a \mathbb{P}^6 (resp. \mathbb{P}^3)-bundle over the product space $\mathbb{P}^2 \times \mathbf{B}(4, 1)$.*
(2) *The locus C_8^+ (resp. C_8^-) at $\alpha = 8$ is a \mathbb{P}^5 (resp. \mathbb{P}^3)-bundle over the product space $\mathbb{P}^2 \times \mathbf{B}(4, 2)$.*

Proof. One can easily check that, if the pair $(1, F_{4m-1})$ (resp. $(1, F_{4m})$) is semistable, so is F_{4m-1} (resp. F_{4m}). Hence the descriptions of the base spaces come from the fact that $\mathbf{M}^\alpha(4, -1) \cong \mathbf{B}(4, 1)$ and $\mathbf{M}^\alpha(4, 0) \cong \mathbf{B}(4, 2)$ for all α . Also, the free resolutions of $F_{d+m+\chi}$ are given in [9].

$$0 \rightarrow 2\mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F_{4m-1} \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3) \rightarrow \mathcal{O} \oplus \mathcal{O}(-1) \rightarrow F_{4m} \rightarrow 0.$$

Using this fact and Lemma 2.1, we obtain that

- $\text{Ext}^1((1, F_{4m-1}), (0, F_{m+3})) \cong \mathbb{C}^7$,
- $\text{Ext}^1((0, F_{m+3}), (1, F_{4m-1})) \cong \mathbb{C}^3$,
- $\text{Ext}^1((1, F_{4m}), (0, F_{m+2})) \cong \mathbb{C}^6$, and
- $\text{Ext}^1((0, F_{m+2}), (1, F_{4m})) \cong \mathbb{C}^3$.

So we have the result in the claim. \square

Recall that the wall types at $\alpha = 3$ are given by

$$(1, (4, 1)) \oplus (0, (1, 1)), (1, (3, 0)) \oplus (0, (2, 2)) \text{ or } (1, (3, 0)) \oplus (0, (1, 1)) \oplus (0, (1, 1)).$$

Since the wall is not simple, we need more detail calculation. Obviously, the first two types are general case. The third one is the intersection part. Let A^+ (resp. A^-) be the locus of the $3 + \epsilon$ (resp. $3 - \epsilon$)-stable pairs whose JH-filtration type is the first one. Let B^+ (resp. B^-) be the locus of the $3 + \epsilon$ (resp. $3 - \epsilon$)-stable pairs whose JH-filtration type is the second one. Let $C_3^+ = A^+ \cup B^+$ and $C_3^- = A^- \cup B^-$. Let D^+ (resp. D^-) be the locus of the $3 + \epsilon$ (resp. $3 - \epsilon$)-stable pairs $(1, F)$ fitting into a non-split exact sequence

$$0 \rightarrow (0, F_{2m+2}) \rightarrow (1, F) \rightarrow (1, F_{3m}) \rightarrow 0$$

$$(\text{resp. } 0 \rightarrow (1, F_{3m}) \rightarrow (1, F) \rightarrow (0, F_{2m+2}) \rightarrow 0)$$

such that $F_{2m+2} = F_{m+1} \oplus F'_{m+1}$. Since the stable pairs in the intersection part may have non-trivial automorphism, we compute the wall crossing separately in Lemma 2.6 and Lemma 2.7.

Lemma 2.6. (1) (a) *The locus A^+ is a \mathbb{P}^4 -bundle over $\mathbb{P}^2 \times \mathbf{B}(4, 3)$. The locus $A^+ \cap D^+$ is a disjoint union of a \mathbb{P}^3 -bundle over a \mathbb{P}^3 -bundle over $(\mathbb{P}^2 \times \mathbb{P}^2 - \Delta) \times \mathbf{B}(3, 0)$ (where Δ is the diagonal of $\mathbb{P}^2 \times \mathbb{P}^2$) and of a \mathbb{P}^2 -bundle over a \mathbb{P}^3 -bundle over $\mathbb{P}^2 \times \mathbf{B}(3, 0)$.*
(b) *The locus $B^+ - A^+$ is a \mathbb{P}^7 -bundle over $\mathbf{M}^+(3, 0) \times \mathbf{M}(2, 2)^s$. Here the space $\mathbf{M}(2, 2)^s$ consists of the stable sheaves which is isomorphic to $\mathbb{P}^5 - V$ where the $V \cong \text{Sym}^2(\mathbb{P}^2)$ is the space of degenerated conics.*

- (2) (a) *The locus A^- is a \mathbb{P}^3 -bundle over $\mathbf{M}(1, 1) \times \mathbf{M}^+(4, 1)$. The locus $A^- \cap D^-$ is a disjoint union of a \mathbb{P}^2 -bundle over a \mathbb{P}^2 -bundle over $(\mathbb{P}^2 \times \mathbb{P}^2 - \Delta) \times \mathbf{B}(3, 0)$ and of a \mathbb{P}^1 -bundle over a \mathbb{P}^2 -bundle over $\mathbb{P}^2 \times \mathbf{B}(3, 0)$.*
- (b) *The locus $B^- - A^-$ is a \mathbb{P}^5 -bundle over $\mathbf{M}^+(3, 0) \times \mathbf{M}(2, 2)^s$.*

Proof. For $\alpha = 3 + \epsilon$, the α -stable pairs $(1, F)$ in A^+ fit into a non-split exact sequence

$$(2.3) \quad 0 \rightarrow (0, F_{m+1}) \rightarrow (1, F) \rightarrow (1, F_{4m+1}) \rightarrow 0.$$

Also one can easily check that all of the non-split extension in the equation above are α -stable. Thus A^+ is a $\mathbb{P}(\text{Ext}^1((1, F_{4m+1}), (0, F_{m+1})))$ -bundle over $\mathbf{M}(1, 1) \times \mathbf{M}^\infty(4, 3)$. Note that $\mathbf{M}^\alpha(4, 1) \cong \mathbf{M}^\infty(4, 1) \cong \mathbf{B}(4, 3)$ for $\alpha > 3$. By direct computation, we know that

$$(2.4) \quad \text{Ext}^1((1, F_{4m+1}), (0, F_{m+1})) \cong \mathbb{C}^5.$$

If $(1, F) \in A^+ \cap D^+$ in (2.3), the pair $(1, F_{4m+1})$ should fit into the exact sequence

$$(2.5) \quad 0 \rightarrow (0, F'_{m+1}) \rightarrow (1, F_{4m+1}) \rightarrow (1, F_{3m}) \rightarrow 0.$$

By the long exact sequence obtained by (2.5), we see

$$\text{Ext}^1((1, F_{4m+1}), (0, F_{m+1})) \xrightarrow{\xi} \text{Ext}^1((0, F'_{m+1}), (0, F_{m+1})) \rightarrow \text{Ext}^2((1, F_{3m}), (0, F_{m+1})).$$

But the last term is $\text{Ext}^2((1, F_{3m}), (0, F_{m+1})) = 0$ because $H^1(F_{m+1}) = 0$ and $\text{Ext}^2(F_{3m}, F_{m+1}) \cong \text{Ext}^0(F_{m+1}, F_{3m}(-3)) = 0$ by the stability of F_{m+1} . That is, the map ξ is surjective. Then the central term $(1, F)$ of the non split extension (2.3) lie in the space D^+ if and only if ξ is zero if applied to the class of (2.3). This is because, by definition, the image of the class of (2.3) in $\text{Ext}^1((1, F_{4m+1}), (0, F_{m+1}))$ by ξ corresponds to the pullback class of (2.3) in $\text{Ext}^1((0, F'_{m+1}), (0, F_{m+1}))$ via the morphism $(0, F'_{m+1}) \hookrightarrow (1, F_{4m+1})$.

But we know that

$$\text{Ext}^1((0, F'_{m+1}), (0, F_{m+1})) = \text{Ext}^1(F'_{m+1}, F_{m+1}) \simeq \begin{cases} \mathbb{C} & \text{if } F'_{m+1} \neq F_{m+1}, \\ \mathbb{C}^2 & \text{if } F'_{m+1} = F_{m+1}. \end{cases}$$

Thus the kernel of ξ depends on the choices of F'_{m+1} and F_{m+1} . Note that the classes of non-split extensions as (2.5) are parameterized by

$$(2.6) \quad \mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F'_{m+1}))) \cong \mathbb{P}^3.$$

Combining with this fact, we get the result (1)-(a).

The stable pairs in $B^+ - A^+$ are supported on a quintic curve with smooth conic as a component. Hence, the locus $B^+ - A^+$ is a $\mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F_{2m+2})))$ -bundle over $\mathbf{M}^+(3, 0) \times \mathbf{M}(2, 2)^s$. By using the resolution of the sheaves, we see that

$$(2.7) \quad \text{Ext}^1((1, F_{3m}), (0, F_{2m+2})) \cong \mathbb{C}^8$$

and so we get (1)-(b).

The proof of the case $\alpha = 3 - \epsilon$ is the same as that of $\alpha = 3 + \epsilon$ except that

- $\text{Ext}^1((0, F_{m+1}), (1, F_{4m+1})) \cong \mathbb{C}^4$,
- $\text{Ext}^1((0, F'_{m+1}), (1, F_{3m})) \cong \mathbb{C}^3$, and
- $\text{Ext}^1((0, F_{2m+2}), (1, F_{3m})) \cong \mathbb{C}^6$.

By replacing these extensions with that of (2.4), (2.6) and (2.7), one can finish the proof of lemma. \square

Lemma 2.7. (1) *The locus D^+ is the disjoint union of a $\mathbb{P}^3 \times \mathbb{P}^3$ -bundle over $\mathbb{P}^9 \times (V - \overline{\Delta})$ and of a $\text{Gr}(2, 4)$ -bundle over $\mathbb{P}^9 \times \overline{\Delta}$. Here, $V = \text{Sym}^2(\mathbb{P}^2)$ and $\overline{\Delta} = \mathbb{P}^2$ is the diagonal of V .*
 (2) *The intersection locus D^- is the disjoint union of a $\mathbb{P}^2 \times \mathbb{P}^2$ -bundle over $\mathbb{P}^9 \times (V - \overline{\Delta})$ and of a $\text{Gr}(2, 3)$ -bundle over $\mathbb{P}^9 \times \overline{\Delta}$.*

Proof. The stable pairs $(s, F) \in D^+$ fit into an exact sequence

$$(2.8) \quad 0 \rightarrow (0, F_{2m+2}) \rightarrow (s, F) \rightarrow (1, F_{3m}) \rightarrow 0,$$

where $F_{2m+2} = F_{m+1} \oplus F'_{m+1}$ by the definition of the locus D^+ . Note that a non-split extension fitting in (2.8) may not be α -stable. Also, the automorphism of the pair $(0, F_{2m+2}) = (0, F_{m+1}) \oplus (0, F'_{m+1})$ varies depending on the choice of F'_{m+1} and F_{m+1} . Thus we handle such a situation by dividing into two cases.

If $F_{m+1} \neq F'_{m+1}$, then one can easily check that the pair (s, F) is α -stable if and only if the class of (2.8) is contained in

$$\text{Ext}^1((1, F_{3m}), (0, F_{2m+2})) - (\text{Ext}^1((1, F_{3m}), (0, F_{m+1})) \cup \text{Ext}^1((1, F_{3m}), (0, F'_{m+1}))).$$

By quotienting out the space $\text{Aut}((0, F_{2m+2})) \cong \mathbb{C}^* \times \mathbb{C}^*$, we see that the space parameterizing the α -stable pairs (s, F) as above is isomorphic to the product space

$$\mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F_{m+1}))) \times \mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F'_{m+1}))).$$

If $F_{m+1} = F'_{m+1}$, then $\text{Ext}^1((1, F_{3m}), (0, F_{2m+2})) \cong \mathbb{C}^2 \otimes \text{Ext}^1((1, F_{3m}), (0, F_{m+1}))$ and $\text{Aut}((0, F_{2m+2})) \cong \text{GL}(2)$ acts on this \mathbb{C}^2 in the standard way. Hence we have

$$\text{Ext}^1((1, F_{3m}), (0, F_{2m+2}))^s / \text{GL}(2) \cong \text{Gr}(2, \text{Ext}^1((1, F_{3m}), (0, F_{m+1}))).$$

Here the superscript “s” means taking extensions corresponding to α -stable pairs. Since $\text{Ext}^1((1, F_{3m}), (0, F_{m+1})) \cong \mathbb{C}^4$, we have proved the second part of item (1). The case $\alpha < 3$ is the same as that of $\alpha > 3$ except that

$$\text{Ext}^1((0, F_{m+1}), (1, F_{3m})) \cong \mathbb{C}^3$$

so we get the results in item (2). \square

Remark 2.8. We remark that the locus satisfying the condition $F_{m+1} \neq F'_{m+1}$ in (1) (similarly in (2)) of the lemma above is not a Zariski locally trivial fibration. The wall crossing locus can be explained in a different way which enable us to compute the virtual Poincaré polynomial (cf. [20]). Recall that $V - \overline{\Delta} \cong (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta) / \mathbb{Z}_2$. Let Z be the projective bundle over $\mathbb{P}^9 \times \mathbb{P}^2$ with fiber $\mathbb{P}(\text{Ext}^1((1, F_{3m}), (0, F_{m+1}))) \cong \mathbb{P}^3$. The bundle Z can be constructed from the tautological pair of the extensions ([25]). Let $p : Z \times_{\mathbb{P}^9} Z \rightarrow \mathbb{P}^9 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the canonical projection. Then the group \mathbb{Z}_2 equivariantly acts on the both spaces. Let us denote the descent map by

$$\bar{p} : Z \times_{\mathbb{P}^9} Z / \mathbb{Z}_2 \rightarrow \mathbb{P}^9 \times (\mathbb{P}^2 \times \mathbb{P}^2 / \mathbb{Z}_2) \cong \mathbb{P}^9 \times V.$$

Then one can easily see that the inverse image $\bar{p}^{-1}(\mathbb{P}^9 \times (V - \overline{\Delta}))$ is exactly the $\mathbb{P}^3 \times \mathbb{P}^3$ -fibration over $\mathbb{P}^9 \times (V - \overline{\Delta})$, which is isomorphic to the quotient space $(Z \times_{\mathbb{P}^9} Z - (p \times_{\mathbb{P}^9} p)^{-1}(\Delta)) / \mathbb{Z}_2$. Applying the formula in [21, Lemma 2.6], one can get the virtual Poincaré polynomial of the later space.

For later use, let us compute the variation of the virtual Poincaré polynomial at the wall $\alpha = 3$.

Corollary 2.9.

(2.9)

$$P(C_3^+) - P(C_3^-) = p^4 + 4p^5 + 13p^6 + 27p^7 + 44p^8 + 57p^9 + 66p^{10} + 70p^{11} + 72p^{12} + 72p^{13} + 72p^{14} + 72p^{15} + 70p^{16} + 66p^{17} + 57p^{18} + 44p^{19} + 27p^{20} + 13p^{21} + 4p^{22} + p^{23}.$$

Proof. The wall crossing terms are a disjoint union of the locally closed subsets. Thus,

$$\begin{aligned} P(C_3^+) - P(C_3^-) &= [P(A^+ - A^+ \cap D^+) - P(A^- - A^- \cap D^-)] \\ &\quad + [P(B^+ - A^+) - P(B^- - A^-)] + [P(D^+) - P(D^-)]. \end{aligned}$$

By the descriptions in Lemma 2.6, Lemma 2.7 and Remark 2.8, we obtain the result. \square

Since the proof of the lemma below is very similar to that of Lemma 2.5, we omit the proof.

Lemma 2.10. *The flipping locus $C_{\frac{1}{2}}^+$ (resp. $C_{\frac{1}{2}}^-$) at $\alpha = \frac{1}{2}$ is a \mathbb{P}^6 (resp. \mathbb{P}^5)-bundle over the product $\mathbb{P}^5 \times \mathbf{B}(3, 1)$.*

In summary, through Lemma 2.3, 2.5, 2.6, 2.7 and Lemma 2.10, the first part of Theorem 1.1 has been proved.

Remark 2.11. The wall crossing loci C_α^+ for each α can be described in a geometric way (cf. [5]). The wall crossing loci are the loci of pairs of seven points, six points, five, four points on a line with a quartic curve at the wall $\alpha = 18, 13, 8, 3$, respectively, and six points on a conic curve with a cubic curve at $\alpha = \frac{1}{2}$.

2.2. Stratification of the moduli space $\mathbf{M}(5, 2)$. In this subsection, we will study the forgetful map $\mathbf{M}^+(5, 2) \longrightarrow \mathbf{M}(5, 2)$, $(s, F) \mapsto F$ by using the stratification of stable sheaves in $\mathbf{M}(5, 2)$ ([4]).

Proof of (2) in Theorem 1.1. By [4, Theorem 1.1], we know that $h^0(F) \leq 3$ for all stable sheaves $F \in \mathbf{M}(5, 2)$. On the other hand, since $(5, 2) = 1$, there exists a universal family of sheaves \mathcal{F} on $\mathbf{M}(5, 2) \times \mathbb{P}^2$ ([16]). Therefore the Proj of the direct image sheaf $p_*\mathcal{F}$ is isomorphic to $\mathbf{M}^+(5, 2)$ and thus the moduli space $\mathbf{M}^+(5, 2)$ is decomposed into locally closed subsets:

- (1) the \mathbb{P}^1 -bundle over $\mathbf{M}(5, 2)_2$ and
- (2) the \mathbb{P}^2 -bundle over $\mathbf{M}(5, 2)_3$

where $\mathbf{M}(5, 2)_k := \{F \in \mathbf{M}(5, 2) | h^0(F) = k\}$. \square

For later use, we compute the Poincaré polynomial of the exceptional locus $\mathbf{M}(5, 2)_3$.

Proposition 2.12. *The (virtual) Poincaré polynomial of the space $\mathbf{M}(5, 2)_3$ is given by*

$$\begin{aligned} &1 + 3p + 8p^2 + 14p^3 + 19p^4 + 21p^5 + 22p^6 + 22p^7 + 22p^8 + 22p^9 + 22p^{10} + 22p^{11} \\ &+ 22p^{12} + 22p^{13} + 22p^{14} + 22p^{15} + 22p^{16} + 22p^{17} + 21p^{18} + 19p^{19} + 14p^{20} + 8p^{21} + 3p^{22} + p^{23}. \end{aligned}$$

Proof. The locus $\mathbf{M}(5, 2)_3$ is isomorphic to the moduli space $\mathbf{M}(5, -2)_1$ by [3, Proposition 4.2.7]. Also, one can easily check that the forgetful map $\xi : \mathbf{M}^+(5, -2) \rightarrow \mathbf{M}(5, -2)$ is injective and onto the space $\mathbf{M}(5, -2)_1$ by using [18, Table 1]. Moreover, the map ξ is a closed embedding since the differential map $\xi_* : \text{Ext}^1((s, F), (s, F)) \rightarrow \text{Ext}^1(F, F)$ is injective by $\text{Hom}(s, H^0(F)/(s)) = 0$ (Lemma 2.1). Therefore,

$$\mathbf{M}^+(5, -2) \cong \mathbf{M}(5, -2)_1 \cong \mathbf{M}(5, 2)_3.$$

Let us compute the polynomial of the moduli space $\mathbf{M}^+(5, -2)$ by using the wall crossings. Among the moduli spaces $\mathbf{M}^\alpha(5, -2)$, one can easily see that there is a single wall crossing at $\alpha = 2$ such that the JH-filtration is given by $(1, (5, -2)) = (1, (4, -2)) \oplus (0, (1, 0))$. Also, by [5, Lemma 2.3], the space $\mathbf{M}^\infty(5, -2)$ is a projective bundle over $\text{Hilb}^3(\mathbb{P}^2)$ with fiber \mathbb{P}^{17} . Since $\text{Ext}^1((1, F_{4m-2}), (0, F_m)) = \text{Ext}^1((0, F_m), (1, F_{4m-2})) = \mathbb{C}^4$, we get

$$P(\mathbf{M}(5, 2)_3) = P(\mathbf{M}^+(5, -2)) = P(\mathbf{M}^\infty(5, -2)) + (P(\mathbb{P}^3) - P(\mathbb{P}^3)) \cdot P(\mathbb{P}^2) \times P(\mathbb{P}^{14}).$$

Also $P(\text{Hilb}^3(\mathbb{P}^2)) = 1 + 2q + 5q^2 + 6q^3 + 5q^4 + 2q^5 + q^6$ ([11]), so the claim is proved. \square

3. BRIDGELAND WALL CROSSING OF THE MODULI SPACE $\mathbf{M}(5, 2)$

In this section, we study the wall crossing of the space $\mathbf{M}(5, 2)$ in the sense of Bridgeland. For the detail of the Bridgeland wall crossing, see [26]. The wall crossing of $\mathbf{M}(5, 2)$ can be done similarly to [6]. So we omit the detail about the wall computations. From now on, we focus on finding the final birational model of $\mathbf{M}(5, 2)$. To solve this, let us describe the ray generator of the effective cone of the moduli space $\mathbf{M}(5, 2)$. As a set, the divisor D is defined as the locus of stable sheaves which is *not* orthogonal to the vector bundle E (for detail, see [26]). The existence of such a vector bundle E has been proved in [26, Theorem 4.3]. Let $A := \phi^* \mathcal{O}(1)$ where the map $\phi : \mathbf{M}(5, 2) \rightarrow |\mathcal{O}_{\mathbb{P}^2}(5)|$ is defined by the Fitting ideal ([16]). Obviously, the divisor A is the nef divisor of the moduli space $\mathbf{M}(5, 2)$.

Lemma 3.1. *The effective cone of $\mathbf{M}(5, 2)$ is generated by the two geometric divisors A and $D = \overline{X_{01}}$. Here the locus X_{01} consists of the stable sheaves of the forms $\mathcal{O}_C(2)(-Z_4 + Z_1)$ such that C is a smooth quintic curve and Z_i is the subscheme of C with length i in a general position.*

Proof. By [26, Theorem 5.3], the divisors A and D generate the rays of the effective cone of the space $\mathbf{M}(5, 2)$. On the other hand, the general free resolution of the stable sheaf $F \in \mathbf{M}(5, 2)$ has two types depending on some algebraic conditions ([18, §2.3]). One can easily check that the sheaf F is orthogonal to E if and only if F fits into the exact sequence $0 \rightarrow \mathcal{T}_{\mathbb{P}^2}(-4) \rightarrow 2\mathcal{O}_{\mathbb{P}^2} \rightarrow F \rightarrow 0$. Hence the complement X_{01} consisting of the stable sheaves $\mathcal{O}_C(2)(-Z_4 + Z_1)$ ([18, Proposition 2.3]) is exactly the support of the divisor D . \square

Proposition 3.2. *The final birational model of the moduli space $\mathbf{M}(5, 2)$ is isomorphic to the Grassmannian variety $\text{Gr}(2, 15)$.*

Proof. From [10, §9.2] and [18, §2.3], the blown-up space \tilde{G} of $\text{Gr}(2, 15)$ along a $\mathbb{P}^2 \times \text{Gr}(2, 6)$ is isomorphic to $\mathbf{M}(5, 2)$ up to codimension one because the exceptional divisor E is supported on the strict transformation of X_{01} in $\mathbf{M}(5, 2)$. Hence $\text{Eff}(\mathbf{M}(5, 2)) = \text{Eff}(\tilde{G})$. Let us compute the corresponding divisor at the

wall W which is right before the collapsing one. Let us denote by λ the map $K(\mathbb{P}^2) \rightarrow \text{Pic}(\mathbf{M}(5, 2))$ which is defined by the Fourier-Mukai transformation (for detail, see [6, 26]). Then $A = \lambda(h^2)$ and $D = \lambda(-5 + 2h + 3h^2)$ for the class $h = [\mathcal{O}_l]$ of a line $l \subset \mathbb{P}^2$ ([26]). Also, the destabilizing objects at the wall W are of type $[\mathcal{O}(-2) \rightarrow 2\mathcal{O}]$ ([18, Table 1]). The same computation as did in [6, Remark 2.12] tells us that $A + D$ is the corresponding divisor at the wall W .

On the other hand, on \tilde{G} ,

$$-15A = K_{\mathbf{M}(5, 2)} = \pi^* K_{\text{Gr}(2, 15)} + 15E.$$

The first equality comes from [26, Lemma 3.1] and the second one comes from [12, Exercice 8.5, II]. Hence $\pi^*(-K_{\text{Gr}(2, 15)}) = 15A + 15D$ is a nef (but not ample) divisor on \tilde{G} because $K_{\text{Gr}(2, 15)}$ is anti-ample and $D = E = \overline{X_{01}}$. Thus the corresponding birational model of the divisors in $[A + D, D]$ is the space $\text{Gr}(2, 15)$. \square

Proposition 3.3. *The moduli space $\mathbf{M}(5, 2)$ can be obtained from the space $\text{Gr}(2, 15)$ by the Bridgeland wall crossings.*

Proof. Proposition 3.2 and [2, Theorem 1.1] imply the statement. \square

Corollary 3.4. *The Poincaré polynomial of the space $\mathbf{M}(5, 2)$ is given by*

$$\begin{aligned} &1 + 2p + 6p^2 + 13p^3 + 26p^4 + 45p^5 + 68p^6 + 87p^7 + 100p^8 + 107p^9 \\ &+ 111p^{10} + 112p^{11} + 113p^{12} + 113p^{13} + 113p^{14} + 112p^{15} + 111p^{16} + 107p^{17} \\ &+ 100p^{18} + 87p^{19} + 68p^{20} + 45p^{21} + 26p^{22} + 13p^{23} + 6p^{24} + 2p^{25} + p^{26}. \end{aligned}$$

Proof. The computation of the wall crossings is similar to that of $\mathbf{M}(6, 1)$ in [6]. So we omit the detail. \square

Remark 3.5. This result is compatible with [27, Theorem 6.1].

4. COMPUTATION OF THE VIRTUAL POINCARÉ POLYNOMIAL OF $\mathbf{M}^\infty(5, 2)$

Summing up the results of the previous sections, we obtain the virtual Poincaré polynomial of $\mathbf{M}^\infty(5, 2)$.

Proof of Corollary 1.3. From part (2) in Theorem 1.1, Corollary 3.4 and Proposition 2.12, we obtain

$$\begin{aligned} P(\mathbf{M}^+(5, 2)) &= (P(\mathbf{M}(5, 2)) - P(\mathbf{M}(5, 2)_3)) \cdot P(\mathbb{P}^1) + P(\mathbf{M}(5, 2)_3) \cdot P(\mathbb{P}^2) \\ &= 1 + 3p + 9p^2 + 22p^3 + 47p^4 + 85p^5 + 132p^6 + 176p^7 + 209p^8 + 229p^9 \\ &+ 240p^{10} + 245p^{11} + 247p^{12} + 248p^{13} + 248p^{14} + 247p^{15} + 245p^{16} + 240p^{17} \\ &+ 229p^{18} + 209p^{19} + 176p^{20} + 132p^{21} + 85p^{22} + 47p^{23} + 22p^{24} + 9p^{25} + 3p^{26} + p^{27}. \end{aligned}$$

Let us add the wall crossing terms in Lemma 2.3, Lemma 2.5, Corollary 2.9 and Lemma 2.10. Let $P(C_\alpha) := P(C_\alpha^+) - P(C_\alpha^-)$. Then,

$$\begin{aligned} P(\mathbf{M}^\infty(5, 2)) &= P(\mathbf{M}^+(5, 2)) + P(C_{18}) + P(C_{13}) + P(C_8) + P(C_3) + P(C_{\frac{1}{2}}) \\ &= P(\mathbf{M}^+(5, 2)) + [P(\mathbb{P}^7)P(\mathbb{P}^2)P(\mathbb{P}^{14}) - P(\mathbb{P}^3)P(\mathbb{P}^2 \times (\mathbb{P}^{14} - \mathbb{P}^9)) - P(\mathbb{P}^4)P(\mathbb{P}^2)P(\mathbb{P}^9)] \\ &+ (P(\mathbb{P}^6) - P(\mathbb{P}^3))P(\mathbb{P}^2)P(\mathbf{B}(4, 1)) + (P(\mathbb{P}^5) - P(\mathbb{P}^3))P(\mathbb{P}^2)P(\mathbf{B}(4, 2)) + (2.9) \\ &+ (P(\mathbb{P}^6) - P(\mathbb{P}^5))P(\mathbb{P}^5)P(\mathbf{B}(3, 1)). \end{aligned}$$

Here, $P(\mathbf{B}(d, 1)) = P(\mathbb{P}^{\frac{d(d+3)-2}{2}}) \cdot P(\mathbb{P}^2)$ for $d = 3, 4$ and $P(\mathbf{B}(4, 2)) = P(\mathbb{P}^{12}) \cdot (1 + 2p + 3p^2 + 2p^3 + p^4)$ ([5, Lemma 2.3] and [11]). The claim is proved with the help of the computer program Maple. \square

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